UNIT - 3

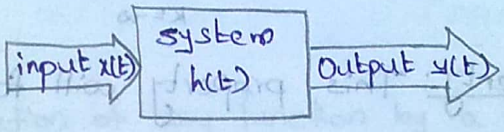
ANALYSIS OF LINEAR SYSTEMS

LINEAR TIME - INVARIANT SYSTEMS MOTIVATION

Continuous and discrete-time systems that are both linear and time-invariant (LTI) play a central role in digital signal processing, communication engineering and control applications:

* Many physical systems are either LTI or approximately so.
* Many efficient tools are available for the analysis and design of LTI systems (e.g. spectral analysis).

Consider the general input-output block diagram of a system. The response of the system $h(t)$ to an input signal $x(t)$ is found by a convolution process, which takes into consideration the complete history of the signal and the information in the system memory.



$$y(t) = \underbrace{x(t) * h(t)}_{\text{Convolution}} = \int_{-\infty}^{\infty} x(u) \cdot h(t-u)\, du$$

Similarly, for the discrete-time case.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

Note that $h[n]$ is called both impulse response and unit pulse response.

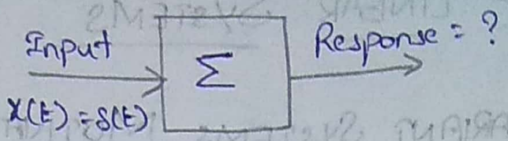
Why impulse response or unit pulse response?

Let the input to the continuous LTI system be $x(t) = \delta(t)$. Then from the definition of the convolution operation we write:

$$y(t) = x(t) * h(t) = \delta(t) * h(t) = \int_{-\infty}^{\infty} \delta(\tau) \cdot h(t-\tau)\, d\tau = h(t)$$

The last integral above is obtained from the shifting theorem.

Impulse response of an accumulator.



$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Properties of linear Convolution.

<u>Commutativity Property</u>:- Convolution is a commutative operation, i.e., the roles of $x(t)$ and $h(t)$ can be interchanged. Similarly, for $x[n]$ and $h[n]$.

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\,d\tau$$

$$y[n] = x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\cdot x[n-k]$$

<u>Associativity Property</u>: This property will form the basis for cascade (series) systems:

$$y(t) = [x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)] \cancel{* x(t) * h_s}$$

$$= x(t) * h_s(t)$$

Where $h_s(t)$ represents the cascade connection of two subsystems $h_1(t)$ and $h_2(t)$, respectively:

$$h_s(t) = h_1(t) * h_2(t)$$

Combined cascade system impulse response $h_s(t)$ is equal to the convolution of system response of individual subsystems. This result can easily be extended to the series connection of many systems via repeated applications of the associativity property.

Distributivity Property: This property, on the other hand, forms the basis for parallel systems.

$$y(t) = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$
$$= x(t) * [h_1(t) + h_2(t)] = x(t) * h_p(t).$$

As in the previous case, $h_p(t)$ corresponds to the parallel combination of two subsystems.

* Linear Convolution Examples (Elementary):

Ex:- Convolution of signals with delta and unit-step functions.

$$\Rightarrow x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\cdot\delta(t-\tau)\, d\tau$$
$$= x(t)$$

$$\Rightarrow x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} \delta(\tau-t_0)\, x(t-\tau)\, d\tau$$
$$= x(t-t_0)$$

$$\Rightarrow x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\cdot u(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{t} x(\tau)\, d\tau$$

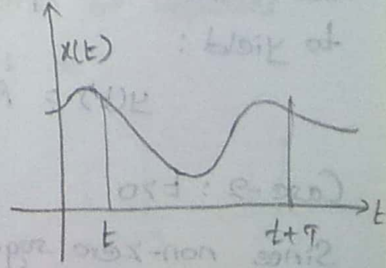
Observation :-

* The convolution of any function by a delta function gives the original function and the convolution of any function with a shifted version of the delta function results in the shifted replica of the original function.
* Convolving a signal by a unit-step function is equivalent to a perfect integrator.

Example : Time averaging

Time averaging is frequently employed in finding average behaviour or mean of systems or signals or data.

$$\overline{x(t)} = x_{ave}(t) = \frac{1}{T}\int_{t}^{t+T} x(\tau)\, d\tau$$



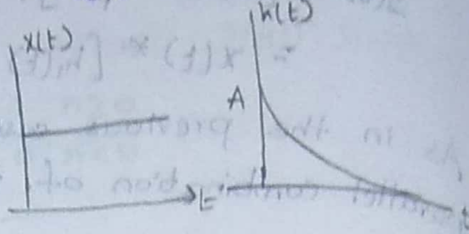
This can be computed in terms of step functions as follows:

$$\overline{x(t)} = \frac{1}{T}\left[\int_{-\infty}^{t+T} x(\tau)\, d\tau - \int_{-\infty}^{t} x(\tau)\, d\tau\right]$$
$$= \frac{1}{T}\left[x(t) * u(t+T) - x(t) * u(t)\right]$$
$$= \frac{1}{T}\, x(t) * [u(t+T) - u(t)]$$

Example: Response of a capacitive circuit to a switched DC voltage, where input and system impulses are simply:

$$x(t) = V \cdot u(t) \text{ and } h(t) = A \cdot e^{-at} \cdot u(t) \text{ where } a > 0.$$

The task is to compute:



$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

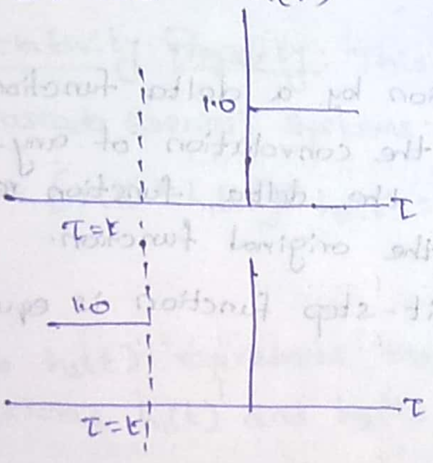
$$= \int_{-\infty}^{\infty} V \cdot A e^{-a(t-\tau)} u(\tau)\, u(t-\tau)\, d\tau$$

Limits of integration are very critical and decided by the non-zero segments of the product of two step functions:

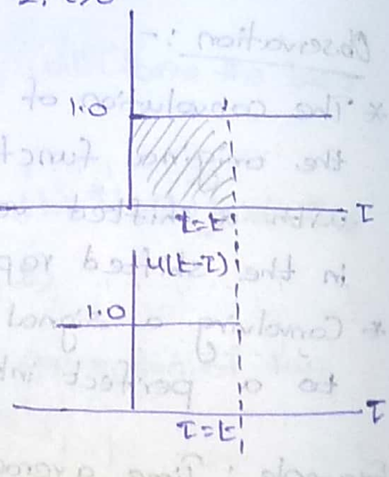
$$u(\tau) \cdot u(t-\tau)$$

Let us now find these non-zero segments with graphical support

Case-1: $t<0$



case 2: $t>0$



Case-1: $t<0$

Since non-zero segments of the product is zero as clearly seen from the plots, the integral in eqn is also zero to yield:

$$y(t) = AV \int_{-\infty}^{\infty} 0 \cdot d\tau = 0.$$

Case-2: $t>0$

Since non-zero segments of the product is the region between 0 and t as shown above, this time the limits of integral that time becomes:

$$y(t) = AV \int_{0}^{t} e^{-a(t-\tau)}\, d\tau$$

$$y(t) = Av\, e^{-at} \int_0^t e^{a\tau}\, d\tau$$

$$= \frac{Av}{a} \cdot e^{-at} \cdot \left. e^{at} \right]_0^t$$

$$= \frac{Av}{a} (1 - e^{-at})$$

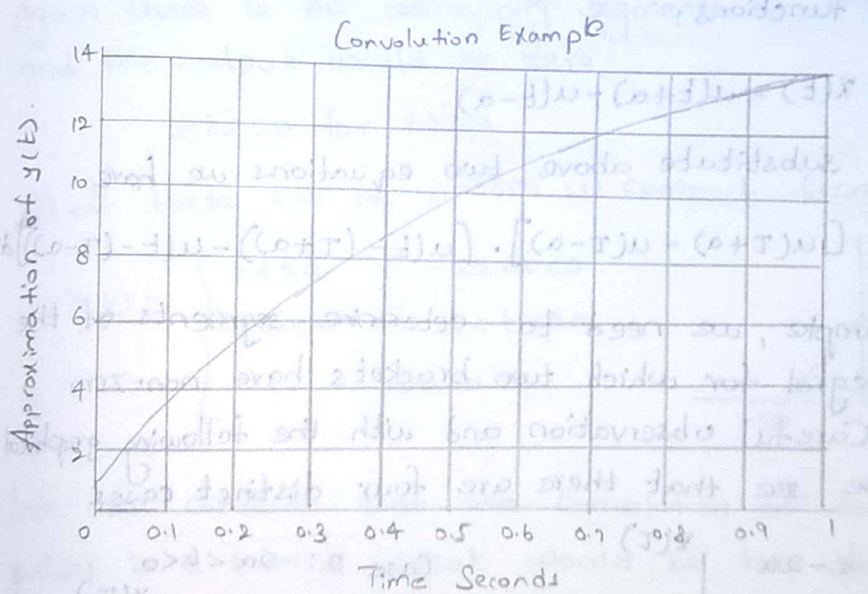
When we combine these two results into a single equation using a unit step function we have the final answer.

$$\frac{Av}{a} (1 - e^{-at}) = 0$$

$$1 - e^{-at} = 0$$

$$e^{-at} = 1$$

```
x = ones (size (t))
y = conv (n, x)
plot (t, y(1:21))
title ("Numerical convolution");
xlabel ("Time, seconds'); ylabel ('Approximation of y(t)')
grid, axis
```



Ex: Convolution of functions with a collection of impulses.

Let the input and the system functions be:

$$x(t) = G_{2a} \left[\frac{t-a}{2a} \right]$$

$$= \text{rect} \left[\frac{t-a}{2a} \right] \text{ and}$$

$$h(t) = \delta(t + 2a) - \delta(t - 2a)$$

$$y(t) = \text{rect}\left[\frac{t-a}{2a}\right] * \delta(t-2a) - \text{rect}\left[\frac{t-a}{2a}\right] * \delta(t-2a)$$

Using the properties of delta functions:

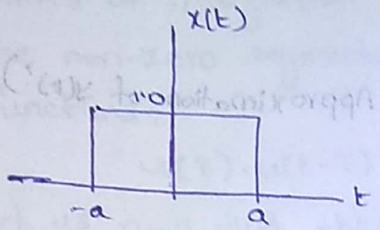
$$y(t) = \text{rect}\left[\frac{(t+2a)-a}{2a}\right] - \text{rect}\left[\frac{(t-2a)-a}{2a}\right]$$

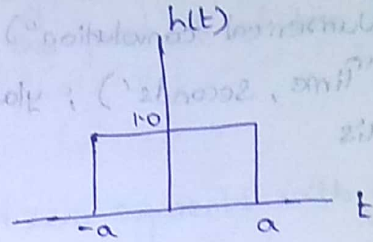
$$= \text{rect}\left[\frac{t+a}{2a}\right] - \text{rect}\left[\frac{t-3a}{2a}\right]$$

First rectangle of length $2a$ is centered at $-a$ and the second one is an inverted rectangle of length $2a$ again but centered at $3a$.

Example: Convolution of two finite duration gate (rectangular) functions.

Task: Evaluate $y(t) = \text{rect}(t/2a) * \text{rect}(t/2a)$





Both of these functions can be represented in terms of unit-step functions:

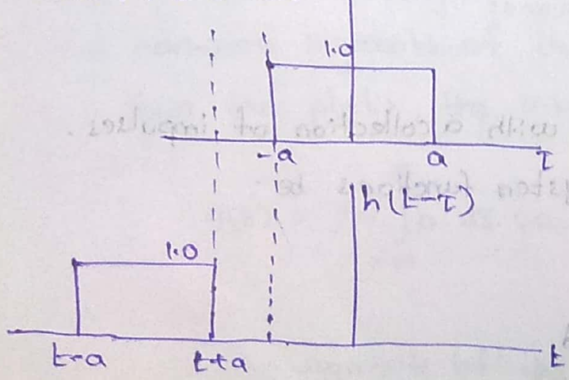
$$x(t) = u(t+a) - u(t-a)$$

when we substitute above two equations we have

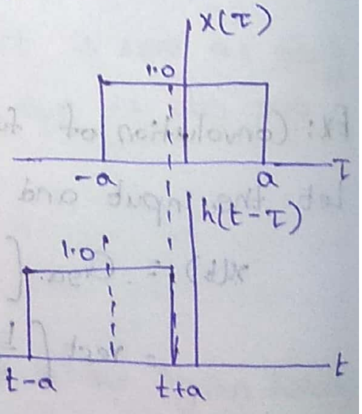
$$y(t) = \int_{-\infty}^{\infty} \{[u(\tau+a) - u(\tau-a)] \cdot [u(t-(\tau+a)) - u(t-(\tau-a))]\} d\tau$$

As in example, we need to determine segments of the above integral for which two brackets have non-zero product. Careful observation and with the following graphical support we see that there are four distinct cases.

Case 1: $t < -2a$



Case 2: $-2a < t < 0$



Case - 1 : $t < -2a$

There is no overlapping segments of the two pulses and the integral in equation would yield zero.

$$y(t) = 0 \text{ for } t < -2a$$

Case - 2 : $-2a < t < 0$

The interval between $-a$ and $t+a$ are common to both pulses then the integral becomes :

$$y(t) = \int_{-a}^{t+a} 1 \cdot d\tau = t + 2a$$

By sliding the lower pulse (the system function) in above figure to the left we get two other cases.

Case - 3 : $0 < t < 2a$

The interval between $t-a$ and a are common to both pulses and we get :

$$y(t) = \int_{t-a}^{a} 1 \, d\tau = 2a - t$$

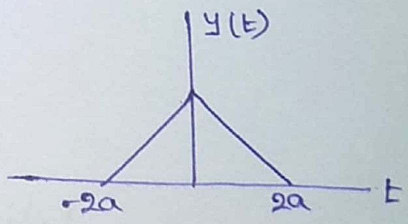
Case - 4 : $t > 2a$

Again there is no overlapping segments of the two pulses and the output would be zero.

$$y(t) = 0 \text{ for } t > 2a$$

All of these can be written in compact form :

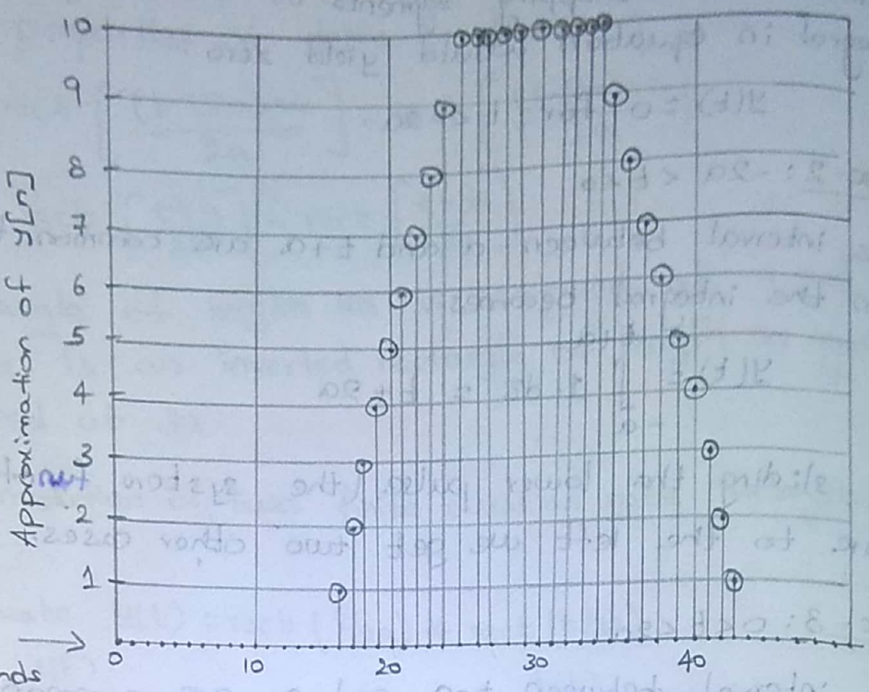
$$y(t) = \begin{cases} t + 2a & \text{if } -2a < t < 0 \\ -t + 2a & \text{if } 0 < t < 2a \\ 0 & \text{otherwise} \end{cases}$$



We can conclude that the convolution of two identical pulses is a triangle. What would be the shape of different size rectangles.

% convolution of decaying exponential

```
n=0:60;
z = zeros ( size (n)); x(6:15)=1;
```

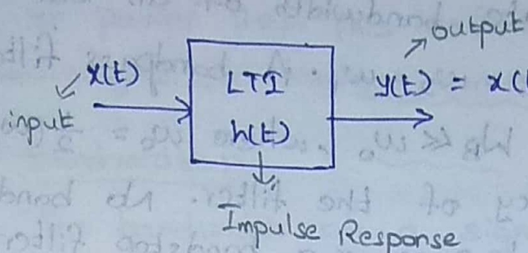



```
n = 0:60;
x = zeros(size(n)); z(6:15)= 1;
h = zeros (size (n)); h (11: 30)= 1;
y = conv(h, x)
Stem (n, y(1:61));
title ('Discrete convolution of Two pulses');
xlabel ('Time, Seconds');
ylabel ('Approximation of y[n]')
grid ; axis
```

Convolution Problems

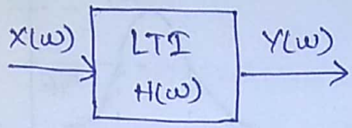
Output of LTI system in frequency domain :-

Time domain :-



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

frequency domain :-



$$Y(\omega) = F\{y(t)\}$$
$$= F\{x(t) * h(t)\}$$
$$= X(\omega) \cdot H(\omega)$$

$$\therefore \boxed{Y(\omega) = X(\omega) \cdot H(\omega)}$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}} \qquad h(t) \neq \frac{y(t)}{x(t)}$$

$H(\omega)$ is called as system function (or) Transfer function.

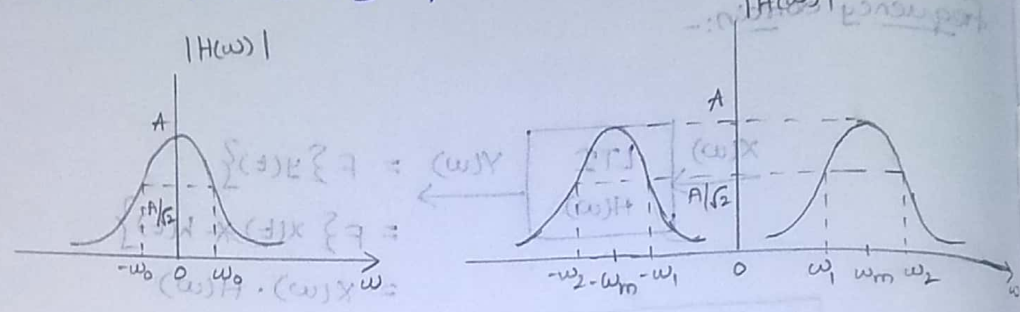
Transfer function of a system $= (H(\omega)) = \dfrac{\text{Frequency domain representation of output}}{\text{Frequency domain representation of input}}$

Bandwidth:- The bandwidth of a signal is defined as the difference between the upper and low frequencies of a signal generated.

A. Filter (or system) Bandwidth :-

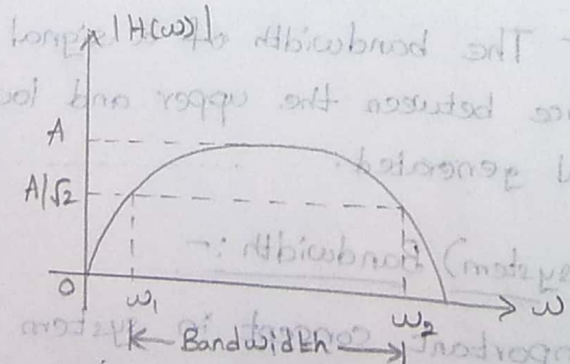
One important concept in system analysis is the bandwidth of an LTI system. There are many different definitions of system bandwidth.

Absolute Bandwidth :- The bandwidth W_B of an ideal low-pass filter equals its cutoff frequency; that is, $W_B = \omega_c$. In this case W_B is called the absolute bandwidth. The absolute bandwidth of an ideal bandpass filter is given by $W_B = \omega_2 - \omega_1$. A bandpass filter is called narrowband if $W_B \ll \omega_0$, where $\omega_0 = \frac{1}{2}(\omega_1 + \omega_2)$ is the center frequency of the filter. No bandwidth is defined for a high-pass or a bandstop filter.



Filter bandwidth

The bandwidth of a system is defined as the range of frequencies over which the magnitude $|H(\omega)|$ remains within $1/\sqrt{2}$ times (within 3 dB) of its value at midband. The bandwidth of a system whose $|H(\omega)|$ plot is shown in figure below is $(\omega_2 - \omega_1)$ where ω_2 is ~~bandwidth of a system whose $|H(\omega)|$~~ called the upper cutoff frequency or upper 3 dB frequency or upper half power frequency and ω_1 is called the lower cutoff frequency or lower 3 dB frequency or lower half power frequency.



Bandwidth

The band limited signals can be transmitted without distortion, if the system bandwidth is atleast equal to the signal bandwidth.

B) Signal Bandwidth :-

The bandwidth of a signal can be defined as the range of positive frequencies in which "most" of the energy or power lies. This definition is rather ambiguous and is subject to various conventions.

3-dB Bandwidth:-

The bandwidth of a signal $x(t)$ can also be defined on a similar basis as a filter bandwidth such as the 3-dB bandwidth, using the magnitude spectrum $|X(\omega)|$ of the signal.

FILTERING

One of the most basic operations in any signal processing system is filtering. Filtering is the process by which the relative amplitudes of the frequency components in a signal are changed or perhaps some frequency components are suppressed. As we saw in the preceding section, for continuous-time LTI system, the spectrum of the output is that of the input multiplied by the frequency response of the system. Therefore, an LTI system acts as a filter on the input signal. Here the word "filter" is used to denote a system that exhibits some sort of frequency-selective behaviour.

Ideal Frequency-Selective Filters:

An ideal frequency-selective filter is one that exactly passes signals at one set of frequencies and completely rejects the rest. The band of frequencies passed by the filter is referred to as the pass band, and the band of frequencies rejected by the filter is called the stop band.

The most common type of ideal frequency-selective filters are the following:

1) Ideal Low-Pass Filter
2) Ideal High-Pass Filter
3) Ideal Bandpass Filter
4) Ideal Bandstop Filter

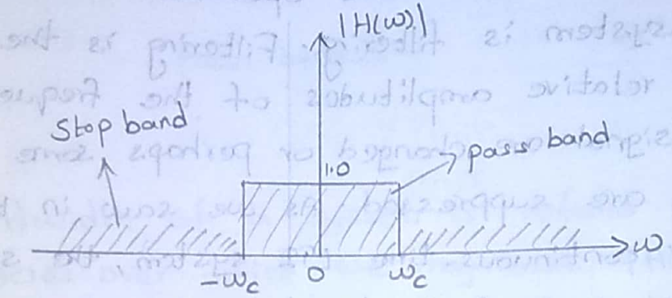
1) **Ideal Low-Pass Filter:-** The filter whose pass band is low frequ

An ideal low-pass filter transmits, without any distortion, all of the signals of frequencies below a certain frequency ω_c radians per second. The signals of frequencies above ω_c radians/second are completely reduced. ω_c is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal low-pass filter (LPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The frequency response characteristics of an ideal LPF are shown in figure. It is a gate function.

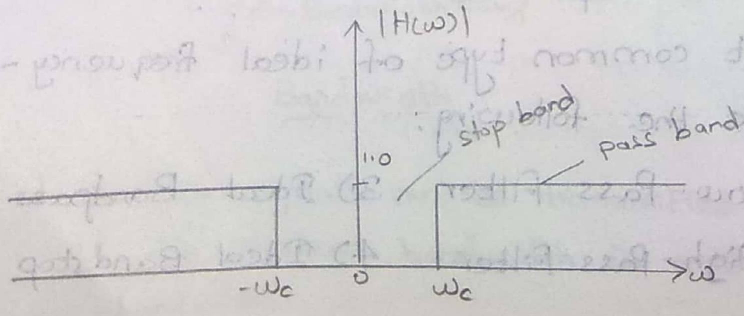


2) **Ideal High-Pass filter:-**

An ideal high-pass filter transmits, without any distortion, all of the signals of frequencies above a certain frequency ω_c radians/second and reduces completely the signals of frequencies below ω_c radians/second, where ω_c is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal high-pass filter (HPF) is specified by

$$|H(\omega)| = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

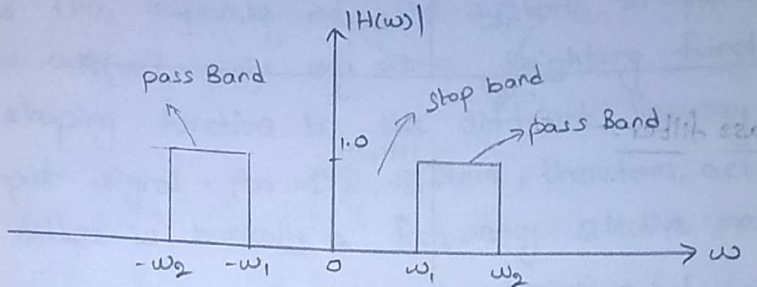


Ideal Band pass Filter:-

An ideal band-pass filter transmits, without any distortion, all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ radians/second and reduces completely the signals of frequencies outside this band. $(\omega_2 - \omega_1)$ is the bandwidth of the band-pass filter. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal Band pass Filter (BPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

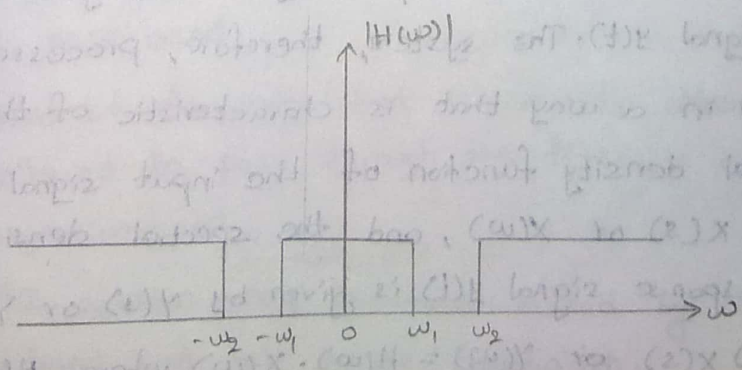


Ideal Bandstop filter (or) ideal band rejection filter:-

An ideal band stop (or) rejection filter rejects totally all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ radians/second and transmits without any distortion all signals of frequencies outside this band. $(\omega_2 - \omega_1)$ is the rejection band. The corresponding phase function for distortionless transmission is $-\omega t_d$.

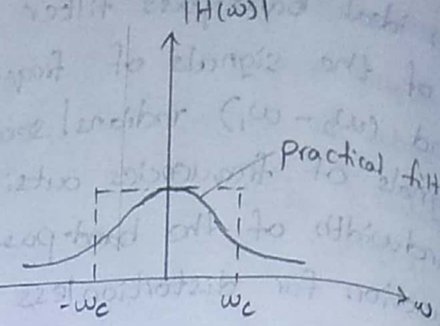
An ideal Band stop (or) rejection filter (BRF) is specified by

$$|H(\omega)| = \begin{cases} 0 & \omega_1 < |\omega| < \omega_2 \\ 1 & \text{otherwise} \end{cases}$$

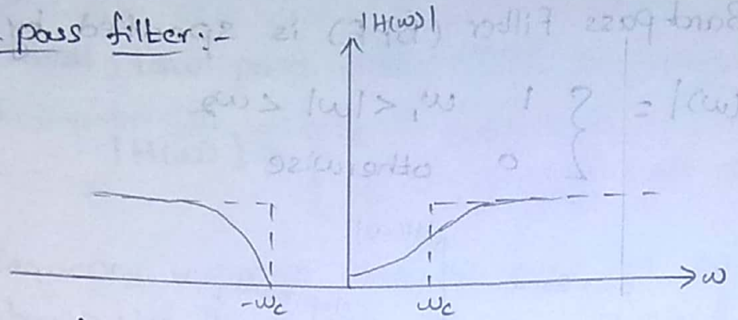


Practical filters :-

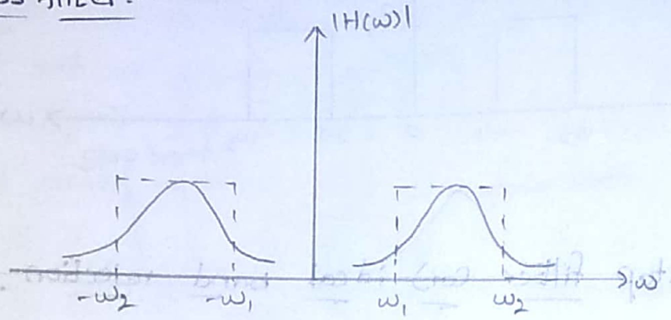
1) Lowpass filter



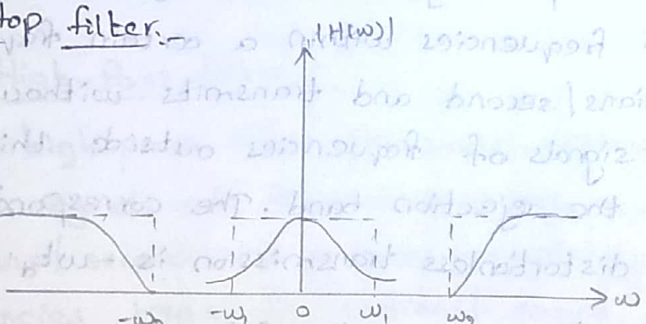
2.) High pass filter :-



3) Band-pass filter :-



4) Band-stop filter :-



Filter Characteristics of linear systems

For a given system an input signal $x(t)$ gives rise to a response signal $y(t)$. The system, therefore, processes the signal $x(t)$ in a way that is characteristic of the system. The spectral density function of the input signal $x(t)$ is given by $X(s)$ or $X(\omega)$, and the spectral density function of the response signal $y(t)$ is given by $Y(s)$ or $Y(\omega)$.

$Y(s) = H(s)\,X(s)$ or $Y(\omega) = H(\omega)\cdot X(\omega)$ where $H(s)$ or $H(\omega)$ is the transfer function (or) system function of the system.

The system, therefore, modifies the spectral density function of the input. The system acts as a kind of filter for various frequency components. Some frequency components are boosted in strength, i.e. they are amplified. Some frequency components are weakened in strength, i.e., they are attenuated (reduced) and some may remain unaffected. Similarly, each frequency component suffers a different amount of phase shift in the process of transmission. The system, therefore, modifies the spectral density function of the input according to its filter characteristics. The modification is carried out according to the transfer function $H(s)$ or $H(\omega)$, which represents the response of the system to various frequency components. $H(\omega)$ acts ~~of~~ as a weighting function or spectral shaping function to the different frequency components in the input signal. An LTI system, therefore, acts as a filter. A filter is basically a frequency selective network.

→ Some LTI systems allow the transmission of only low frequency components and stop all high frequency components. They are called low-pass filters (LPFs)

→ Some LTI systems allow the transmission of only high frequency components and stop all low frequency components. They are called high-pass filters (HPFs)

→ Some LTI systems allow transmission of only a particular band of frequencies and stop all other frequency components. They are called band-pass filters (BPFs)

→ Some LTI systems reject only a particular band of frequencies and allow all other frequency components. They are called band-rejection filters (BRFs).

* The band of frequency that is allowed by the filter is called pass-band.

* The band of frequency that is severely attenuated and not allowed to pass through the filter is called stop-band or rejection-band.

* An LTI system may be characterised by its pass-band, stop-band and half power bandwidth.

The frequency response of continuous-Time LTI systems

Frequency Response:-

The output $y(t)$ of a continuous-time LTI system equals the convolution of the input $x(t)$ with the impulse response $h(t)$; that is,

$$y(t) = x(t) * h(t) \quad -1$$

Applying the convolution property, we obtain

$$Y(\omega) = X(\omega)\, H(\omega) \quad -2$$

where $Y(\omega)$, $X(\omega)$ and $H(\omega)$ are the Fourier transforms of $y(t)$, $x(t)$ and $h(t)$, respectively.

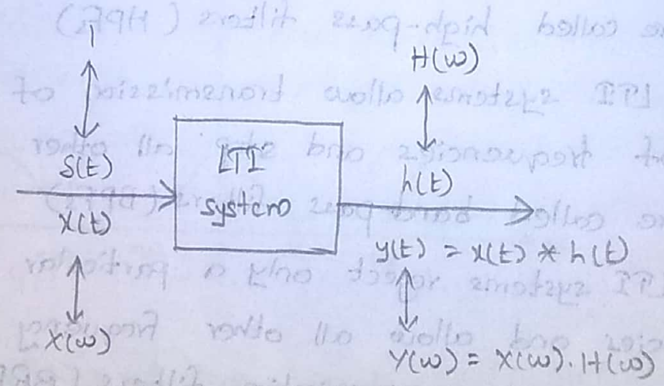
We have

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \quad -3$$

The function $H(\omega)$ is called the frequency response of the system. Relationships represented by 1 & 2 are depicted in below fig

$$H(\omega) = |H(\omega)|\, e^{j\theta_H(\omega)} \quad -4$$

Then $|H(\omega)|$ is called the magnitude response of the system, and $\theta_H(\omega)$ the phase response of the system.



Relationships between inputs and outputs in an LTI system

Consider the complex exponential signal

$$x(t) = e^{j\omega_0 t} \quad -5$$

with Fourier transform

$$X(\omega) = 2\pi\,\delta(\omega - \omega_0) \quad -6$$

Then we have

$$Y(\omega) = 2\pi\, H(\omega_0)\, \delta(\omega - \omega_0) \quad -7$$

Taking the inverse fourier transform of $Y(\omega)$, we obtain

$$y(t) = H(\omega_0)\, e^{j\omega_0 t} \quad - 8$$

which indicates that the complex exponential signal $e^{j\omega_0 t}$ is an eigenfunction of the LTI system with corresponding eignvalue $H(\omega_0)$. Furthermore, by the linearity property, if the input $x(t)$ is periodic with the fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

then the corresponding output $y(t)$ is also periodic with the fourier series.

Missing

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

and using eq-(2), the corresponding output $y(t)$ can be expressed as

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \cdot X(\omega) e^{j\omega t} d\omega$$

Thus, the behavior of a continuous-time LTI system in the frequency domain is completely characterized by its frequency response $H(\omega)$. Let

$$X(\omega) = |X(\omega)|\, e^{j\theta_X(\omega)} \qquad Y(\omega) = |Y(\omega)|\, e^{j\theta_Y(\omega)}$$

Then from eq-(2) we have

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)|$$

$$\theta_Y(\omega) = \theta_X(\omega) + \theta_H(\omega)$$

Hence, the magnitude spectrum $|X(\omega)|$ of the input is multiplied by the magnitude response $|H(\omega)|$ of the system to determine the magnitude spectrum $|Y(\omega)|$ of the output, and the phase response $\theta_H(\omega)$ is added to the phase spectrum $\theta_X(\omega)$ of the input to produce the phase spectrum $\theta_Y(\omega)$ of the output. The magnitude response $|H(\omega)|$ is sometimes referred to as the gain of the system.

Distortionless Transmission

In several applications, such as signal amplification or message signal transmission over a communication channel, we require that the output waveform be a replica of the input waveform. In such cases we need to minimize the distortion caused by the amplifier or the communication channel. It is, therefore, of practical interest to determine the characteristics of a system that allows a signal to pass without distortion (distortionless transmission).

Transmission is said to be distortionless if the input and the output have identical wave shapes within a multiplicative constant. A delayed output that retains the input waveform is also considered to be distortionless. Thus, in distortionless transmission, the input $x(t)$ and the output $y(t)$ satisfy the condition

$$y(t) = G_0 x(t - t_d)$$

The fourier transform of this equation yields

$$F\{y(t)\} = F\{G_0 x(t-t_d)\}$$

$$Y(\omega) = G_0 X(\omega) e^{-j\omega t_d} \quad — ①$$

But

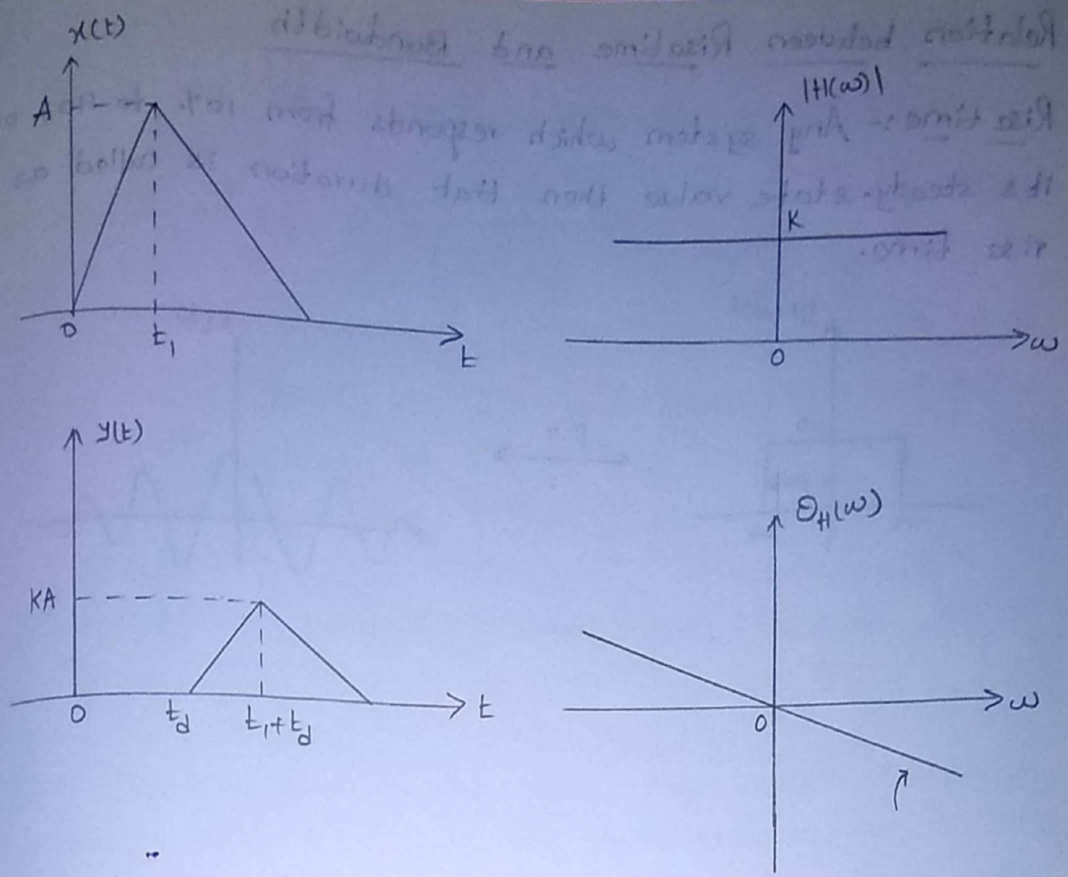
$$Y(\omega) = X(\omega) \cdot H(\omega) \quad — ②$$

Therefore $H(\omega) = G_0 e^{-j\omega t_d}$ — ④ $\qquad H(\omega) = |H(\omega)| e^{j\angle H(\omega)}$ — ③

This is the frequency response required of a system for distortionless transmission. From this equation, it follows that

Compare ③ & ④ $|H(\omega)| = G_0$ and $\angle H(\omega) = -\omega t_d$

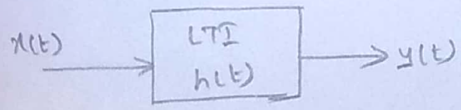
This result shows that for distortionless transmission, the amplitude response $|H(\omega)|$ must be a constant, and the phase response $\angle H(\omega)$ must be a linear function of ω with slope $-t_d$, where t_d is the delay of the output with respect to input.



Causality and poly-wiener criterion for physical realization

A system is said to be causal if it does not produce an output before the input is applied. For an LTI system to be causal, the condition to be satisfied is its impulse response must be zero for t less than zero, i.e.,

$$h(t) = 0 \quad \text{for } t<0$$



$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$\boxed{\int |H(\omega)|^2 d\omega < \infty}$$

→ In the time domain the condition for causality is

$h(t) = 0$

→ To satisfy this in frequency domain the transfer function or system function should be square integrable.

Relation between Risetime and Bandwidth

Rise time:- Any system which responds from 10% to 90% of its steady-state value then that duration is called as rise time.